

Final state interaction in $K \rightarrow 2\pi$ decay

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Abstract

Contrary to wide-spread opinion that the final state interaction (FSI) enhances the amplitude $\langle 2\pi; I = 0 | K^0 \rangle$, we argue that FSI does not increase the absolute value of this amplitude.

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The essential progress in understanding the nature of the $\Delta I = 1/2$ rule in $K \rightarrow 2\pi$ decays was achieved in the paper [1], where the authors had found a considerable increase of contribution of the operators containing a product of the left-handed and right-handed quark currents generated by the diagrams called later the penguin ones. But for a quantitative agreement with the experimental data, a search for some additional enhancement of the $\langle 2\pi; I = 0 | K^0 \rangle$ amplitude produced by long-distance effects was utterly desirable. A necessity of additional enhancement of this amplitude due to long-distance strong interactions was also noted later in [2].

The attempts to take into account the long-distance effects were undertaken in [3] - [15].

In [3], the necessary increase of the amplitude $\langle 2\pi; I = 0 | K^0 \rangle$ was associated with $1/N$ corrections calculated within the large- N approach (N being the number of colours).

In [4], [5], the strengthening of the $\langle 2\pi; I = 0 | K \rangle$ amplitude arised due to a small mass of the intermediate scalar σ meson.

One more mechanism of enhancement of the $\langle 2\pi; I = 0 | K^0 \rangle$ amplitude was ascribed to the final state interaction of the pions [6] - [14]. But as it will be shown in present paper, unitarization of the $K \rightarrow 2\pi$ amplitude in presence of FSI leads to the opposite effect: a decrease of the $\langle 2\pi; I =$

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$0|K^0 >$ amplitude. The similar conclusion had been obtained formerly in [15] where a different from our approach to investigation of FSI effect was used. We exploit the technique based on the effective $\Delta S = 1$ non-leptonic Lagrangian [1]

$$L^{\text{weak}} = \sqrt{2}G_F \sin \theta_C \cos \theta_C \sum_i c_i O_i. \quad (1)$$

Here O_i are the four-quark operators and c_i are the Wilson coefficients calculated taking into account renormalization effect produced by strong quark-gluon interaction at short distances. Using also the recipe for bosonization of the diquark compositions proposed in [2], one obtains the following result:

$$< \pi^+(p_+), \pi^-(p_-); I = 0 | K^0(q) > = \kappa^{(0)}(q^2 - p_-^2), \quad (2)$$

where $\kappa^{(0)}$ is a function of G_F, F_π, θ_C and some combination of c_i . The numerical values of $\kappa^{(0)}$ obtained in [1] and [2] turned out to be insufficient for a reproduction of the observed magnitude of the $< 2\pi; I = 0 | K^0 >$ amplitude.

Could a rescattering the final pions occuring at long distances change the situation? To answer this question, we consider at first the elastic $\pi\pi$ scattering itself.

The elastic $\pi\pi$ scattering.

The general form of the amplitude of elastic $\pi\pi$ scattering is

$$T = < \pi_k(p'_1) \pi_l(p'_2) | \pi_i(p_1) \pi_j(p_2) > = A \delta_{ij} \delta_{kl} + B \delta_{ik} \delta_{jl} + C \delta_{il} \delta_{jk}, \quad (3)$$

where k, l, i, j are the isotopical indices and A, B, C are the functions of $s = (p_1 + p_2)^2, t = (p_1 - p'_1)^2, u = (p_1 - p'_2)^2$.

The amplitudes with the fixed isospin $I = 0, 1, 2$ are

$$T^{(0)} = 3A + B + C, \quad T^{(1)} = B - C, \quad T^{(2)} = B + C. \quad (4)$$

To understand the problems arising in description of $\pi\pi$ scattering in the framework of field theory, let's consider the simplest chiral σ model, where

$$A^{\text{tree}} = \frac{g_{\sigma\pi\pi}^2}{m_\sigma^2 - s} - \frac{g_{\sigma\pi\pi}^2}{m_\sigma^2 - m_\pi^2} = \frac{g_{\sigma\pi\pi}^2}{m_\sigma^2 - m_\pi^2} \cdot \frac{s - m_\pi^2}{m_\sigma^2 - s} \quad (5)$$

and B and C are obtained from A by replacement $s \rightarrow t$ and $s \rightarrow u$, respectively.

It follows from Eqs.(4) and (5), that the isosinglet amplitude $T_{\text{tree}}^{(0)}$ is a sum of the resonance part

$$A_{\text{Res}}^{\text{tree}} = 3A^{\text{tree}} \quad (6)$$

and the potential part

$$A_{\text{Pot}}^{\text{tree}} = B^{\text{tree}} + C^{\text{tree}}. \quad (7)$$

The resonance part must be unitarized summing up the chains of pion loops, that is , taking into account the repeated rescattering of the final pions.

At the one loop order

$$A_{\text{Res}}^{\text{one-loop}} = A_{\text{Res}}^{\text{tree}}(1 + \Re\Pi_R + i\Im\Pi) = A_{\text{Res}}^{\text{tree}}(1 + \Re\Pi_R + i\frac{A_{\text{Res}}^{\text{tree}}\sqrt{1 - 4m_\pi^2/s}}{16\pi}), \quad (8)$$

where $\Re\Pi_R$ is the renormalized real part of the closed pion loop [16]

$$\Re\Pi_R(s) = \Re\Pi(s) - \Re\Pi(m_\sigma^2) - \frac{\partial\Re\Pi(s)}{\partial s}\bigg|_{s=m_\sigma^2}(s - m_\sigma^2). \quad (9)$$

The last two terms in r.h.s. of this equation are absorbed in renormalization of the resonance mass and coupling constant $g_{\sigma\pi\pi}$. In leading order of perturbation theory in $g_{\sigma\pi\pi}$ one has (see also [17])

$$\Re\Pi_R(s) = \frac{A_{\text{res}}^{\text{tree}}(s)}{16\pi^2}\sqrt{1 - 4m_\pi^2/s}\ln\frac{1 - \sqrt{1 - 4m_\pi^2/s}}{1 + \sqrt{1 - 4m_\pi^2/s}}. \quad (10)$$

But in view of very big value of this constant such a calculation does not give a proper estimate of $\Re\Pi_R(s)$. It will be explained below, how to get a reliable magnitude of $\Re\Pi_R(s)$.

The unitarized expression for A_{Res} is ²

$$A_{\text{Res}}^{\text{unitar}} = \frac{A_{\text{Res}}^{\text{tree}}(s)}{1 - \Re\Pi_R(s) - i\Im\Pi_{\text{Res}}} = \frac{A_{\text{Res}}^{\text{tree}}(s)}{1 - \Re\Pi_R(s)} \cdot \frac{1}{1 - i\tan\delta_{\text{Res}}}, \quad (11)$$

where

$$\tan\delta_{\text{Res}} = \frac{A_{\text{Res}}^{\text{tree}}(s)\sqrt{1 - 4m_\pi^2/s}}{16\pi(1 - \Re\Pi_R(s))}. \quad (12)$$

²Strictly speaking, the 4π intermediate state brings a correction in Eq.(11). But its contribution to $\Im\Pi(s)$ is equal to zero because $4m_\pi > m_K$. As for $\Re\Pi_R(s)$, in our approach, all separate contributions to it will be taken into account phenomenologically introducing a form factor, see below.

The Eq.(11) may be rewritten in the form

$$A_{\text{Res}}^{\text{unitar}} = \frac{16\pi \sin \delta_{\text{Res}} e^{i\delta_{\text{Res}}}}{\sqrt{1 - 4m_\pi^2/s}}, \quad (13)$$

leading to the cross section

$$\sigma_{\text{Res}} = \frac{4\pi \sin^2 \delta_{\text{Res}}}{k^2}, \quad k = \frac{\sqrt{s}}{2} \cdot \sqrt{1 - 4m_\pi^2/s}. \quad (14)$$

Of course, the amplitude $T^{(0)}$ must be unitarized including the potential part $B + C$ too. But if this potential part is considerably smaller than the resonance one, the effect of FSI can be estimated roughly from $A_{\text{Res}}^{\text{unitar}}$. To understand what gives the unitarization of $A_{\text{Pot}}^{\text{tree}}$, we use the form of the S matrix of elastic scattering with the total phase shift as a sum of the phase shifts produced by separate mechanisms of scattering [18]. In other words, if there is a number of resonances and if, in addition, there is potential scattering, the matrix S looks as

$$S = e^{2i\delta_{\text{Res}1}} e^{2i\delta_{\text{Res}2}} \dots e^{2i\delta_{\text{Pot}}}. \quad (15)$$

Then, in terms of

$$\delta_{\text{Res}} = \sum_j \delta_{\text{Res}j} \quad \text{and} \quad \delta_{\text{tot}} = \delta_{\text{Res}} + \delta_{\text{Pot}}, \quad (16)$$

$$A^{\text{unitar}} = \frac{16\pi}{\sqrt{1 - 4m_\pi^2/s}} \sin \delta_{\text{tot}} e^{i\delta_{\text{tot}}} \quad (17)$$

or

$$A^{\text{unitar}} = \frac{16\pi}{\sqrt{1 - 4m_\pi^2/s}} (\sin \delta_{\text{Res}} \cos \delta_{\text{Pot}} + \sin \delta_{\text{Pot}} \cos \delta_{\text{Res}}) e^{i\delta_{\text{tot}}}. \quad (18)$$

The phase shifts δ_{Res} and δ_{Pot} can be taken from [19], where the Resonance Chiral Theory of $\pi\pi$ Scattering (RChT) was elaborated. This model incorporates two σ mesons, $f_0(980)$, $\rho(750)$ and $f_2(1270)$. In addition, some phenomenological form factors were introduced in the vertices $\sigma\pi\pi$, $\rho\pi\pi$, $f_2\pi\pi$. Their appearance follows in the field theory from the result (11), according to which the effect of $\Re\Pi_R(s)$ may be incorporated in $g_{\sigma\pi\pi}^2(s)$, where

$$g_{\sigma\pi\pi}^2(s) = \frac{g_{\sigma\pi\pi}^2}{1 - \Re\Pi_R(s)} = g_{\sigma\pi\pi}^2 F(s). \quad (19)$$

The RChT gives a quite satisfactory description of the observed behavior of the phase shifts $\delta_0^0(s)$, $\delta_0^2(s)$, $\delta_1^1(s)$ in the range $4m_\pi^2 \leq s \leq 1 \text{ (GeV)}^2$. For δ_0^0 and δ_0^2 at $s = m_K^2$ we have

$$\delta_0^0 = 36.74^\circ, \quad \delta_0^2 = -8.1^\circ \quad (20)$$

in agreement with the experimental values $\delta_0^0 = 37^\circ \pm 3^\circ$, $\delta_0^2 = -7^\circ \pm 1^\circ$ (see [14]). The phase shifts $\delta_2^0(s)$ and $\delta_2^2(s)$ turn out to be consistent with the results obtained using the Roy's dispersion relations.

A good fit to experimental data on δ_0^0 and δ_0^2 was obtained in [19] with

$$F(q^2) = \exp\left(-0.5(q^2 - m_\pi^2)/(\text{GeV})^2\right). \quad (21)$$

Then

$$\Re \Pi_R(s = m_K^2) = -0.12 \quad (22)$$

and has the same sign as in Eq.(10), but turns out to be considerably smaller in absolute value. For $\sqrt{s} = m_K$, the phase shifts obtained in [19] are³

$$\delta_{\text{Res}} = 46.71^\circ, \quad \delta_{\text{Pot}} = -9.40^\circ. \quad (23)$$

Then

$$\frac{A_{\text{Pot}}^{\text{unitar}}}{A_{\text{Res}}^{\text{unitar}}} = \frac{\sin \delta_{\text{Pot}} \cos \delta_{\text{Res}}}{\sin \delta_{\text{Res}} \cos \delta_{\text{Pot}}} = -0.156. \quad (24)$$

Therefore, the amplitude $A_{\text{Pot}}^{\text{unitar}}$ is small and may be neglected in a rough estimate of FSI effect.

FSI in $K^0 \rightarrow 2\pi$ decay.

Basing on the result (24), we shall estimate effects of FSI in the $K \rightarrow 2\pi$ amplitude, taking into account only the resonance rescattering effect. Then, in one loop approximation, the amplitude (2) is

$$\begin{aligned} & \langle \pi^+(p_+) \pi^-(p_-); I=0 | K^0(q) \rangle_{\text{Res}}^{\text{one-loop}} = \\ & \kappa^{(0)} \left[(q^2 - p_-^2) + \frac{A_{\text{Res}}^{\text{tree}}(q^2)}{(2\pi)^4 i} \int \frac{(q^2 - p^2) d^n p}{[(p-q)^2 - m_\pi^2][p^2 - m_\pi^2]} + i \frac{A_{\text{Res}}^{\text{tree}}(q^2)}{16\pi} (q^2 - p_-^2) \sqrt{1 - 4m_\pi^2/q^2} \right]. \end{aligned} \quad (25)$$

³A difference 1.3° between δ_{Pot} and δ_0^2 is caused by intermediate ρ meson giving the contributions to these phase shifts which are different in value and sign [19].

In the t'Hooft-Veltman scheme of dimensional regularization [20]

$$\begin{aligned} \frac{1}{(2\pi)^4 i} \int \frac{d^n p}{[(p-q)^2 - m^2][p^2 - m^2]} &= \frac{1}{16\pi^2} \left(\ln \frac{M_0^2}{m^2} + 2 + \sqrt{1 - 4m^2/q^2} \ln \frac{1 - \sqrt{1 - 4m^2/q^2}}{1 + \sqrt{1 - 4m^2/q^2}} \right); \\ \frac{1}{(2\pi)^4} \int \frac{p^2 d^n p}{[(p-q)^2 - m^2][p^2 - m^2]} &= \frac{m^2}{16\pi^2} \left(2 \ln \frac{M_0^2}{m^2} + 3 + \sqrt{1 - 4m^2/q^2} \ln \frac{1 - \sqrt{1 - 4m^2/q^2}}{1 + \sqrt{1 - 4m^2/q^2}} \right) \\ M_0 &\rightarrow \infty. \end{aligned} \quad (26)$$

After renormalization excluding the parts of these integrals independent of the external momentum, we come to

$$\begin{aligned} < \pi^+ \pi^- | K^0(q) >_{\text{on-mass-shell}}^{\text{one-loop}} = \\ = \kappa(m_K^2 - m_\pi^2) \left[1 + \frac{A_{\text{Res}}^{\text{tree}}(s)}{16\pi^2} \sqrt{1 - 4m_\pi^2/s} \ln \frac{1 - \sqrt{1 - 4m_\pi^2/s}}{1 + \sqrt{1 - 4m_\pi^2/s}} + i \frac{A_{\text{Res}}^{\text{tree}}(s)}{16\pi} \sqrt{1 - 4m_\pi^2/s} \right]. \end{aligned} \quad (27)$$

This result agrees with the Cabibbo–Gell-Mann theorem [21], according to which the $K \rightarrow 2\pi$ amplitude vanishes in the limit of exact $SU(3)$ symmetry.

The unitarization of the amplitude (27) done in accordance with the prescription (11) leads to the result

$$| < \pi\pi; I=0 | K^0(q^2 = m_K^2) > |_{\text{Res}} = \kappa^{(0)}(m_K^2 - m_\pi^2) \frac{\cos \delta_{\text{Res}}}{1 - \Re \Pi_R(m_K^2)}, \quad (28)$$

where $\Re \Pi_R(m_K^2)$ is determined by Eq.(22). In accordance with Eq.(12), the factor characterising FSI effect can be expressed through δ_{Res} and $A_{\text{Res}}^{\text{tree}}$:

$$\chi^{(0)} = \frac{\cos \delta_{\text{Res}}(s = m_K^2)}{1 - \Re \Pi_R(m_K^2)} = \frac{16\pi \sin \delta_{\text{Res}}(s = m_K^2)}{A_{\text{Res}}^{\text{tree}}(s = m_K^2) \sqrt{1 - 4m_\pi^2/m_K^2}}. \quad (29)$$

The phase shift δ_{Res} obtained in the framework of the theory [19] and given by Eq.(23) is compatible with the data giving $\delta_{\text{Res}}^0 \approx (\delta_0^0 - \delta_0^2)^{\text{exp}} = 44.0^\circ \pm 3.2^\circ$.

A value of $A_{\text{Res}}^{\text{tree}}$ evidently depends on the used σ model which, besides a good description of data on the phase shifts, must reproduce the other experimental observations. In particular, the result for a mass of the lightest σ meson: $m_\sigma \leq 700$ MeV [22]. An existence of σ particle with mass $m_\sigma = 600 \div 700$ MeV explains also why the observed value of the form factor $F_1(Q^2 = 4m_\pi^2)$ of K_{e4} decay is by 50% bigger than a value predicted in the non-physical point $Q^2 = m_\pi^2$ by the theory based on algebra of currents and soft-pion technics [23]⁴.

⁴In the region $m_\pi^2 \geq Q^2 \leq 4m_\pi^2$ the amplitude is purely real

Using the formulae represented in [23] and in Appendix in [24] for the coupling constants and masses of σ mesons in $U(3)_L \otimes U(3)_R$ symmetric meson theory, one can obtain that at $m_\sigma \leq 700$ MeV the normalized amplitude

$$A_{\text{Res}}^{\text{tree}} = \frac{3}{2}(s - m_\pi^2) \left[\frac{g_{\sigma_1 \pi \pi}^2}{(m_{\sigma_1}^2 - m_\pi^2)(m_{\sigma_1}^2 - s)} + \frac{g_{\sigma_2 \pi \pi}^2}{(m_{\sigma_2}^2 - m_\pi^2)(m_{\sigma_2}^2 - s)} \right] \geq 72. \quad (30)$$

In the theory [19] $m_{\sigma_1} = 697.6$ MeV and $A_{\text{Res}}^{\text{tree}} = 72.4$. Therefore, using the data on the phase shifts, one obtains

$$\chi^{(0)} = 0.585_{-0.024}^{+0.033}. \quad (31)$$

This value is compatible with $\chi^{(0)} = 0.61$ given by Eq.(28) where $\Re \Pi_R(s = m_K^2) = -0.12$. The part connected with the potential rescattering, being small, can not change the conclusion that FSI diminishes the $< 2\pi; I = 0 | K^0 >$ amplitude.

In the case of the $< 2\pi; I = 2 | K^0 >$ amplitude, FSI adds to the initial expression $\kappa^{(2)}(m_K^2 - m_\pi^2)$ the factor

$$\chi^{(2)} = 16\pi \sin \delta_0^2 e^{i\delta_0^2} / (A_2^{\text{tree}} \sqrt{1 - 4m_\pi^2/m_K^2}) \quad (32)$$

At the used in [19] parameters, the normalized $A_2^{\text{tree}} = -8.293$, $\delta_0^2 = -8.1^\circ$ and as a result $\chi^{(2)} = 1.035$.

The influence of FSI on the $K^0 \rightarrow 2\pi$ decay was studied in the framework of σ model in [25]. In this paper, the authors, however, put $\Re \Pi = 0$. Then

$$A^{\text{unitar}} = A^{\text{tree}} / (1 - i\Im \Pi)$$

and this formula was used by them to estimate the FSI effects in the $K^0 \rightarrow 2\pi$ decay. But earlier the same authors had found that $\Re \Pi \neq 0$ [17]. In this case, the unitarization leads to A^{unitar} for the elastic $\pi\pi$ scattering given in Eq.(11) and to $A_{\text{Res}}^{\text{unitar}}(K \rightarrow 2\pi; I = 0)$ in Eq.(28). As it is seen from Eq.(28), FSI could increase or diminish the $K \rightarrow 2\pi$ amplitude depending on relative magnitudes of $\cos \delta$ and $(1 - \Re \Pi_R)$. We have shown that $\cos \delta / (1 - \Re \Pi_R) < 1$, that allows us to affirm that FSI diminishes the isosinglet part of the $K \rightarrow 2\pi$ amplitude.

Conclusion.

Carring out our investigation in the framework of field theory with the phenomenological mesonic Lagrangian we have not found an enhancement of the

amplitude $\langle \pi\pi; I=0 | K^0 \rangle$ due to final state interaction of pions. On the contrary, our analysis has shown that FSI diminishes this amplitude. Hence, FSI is not at all the mechanism bringing us nearer to explanation of the $\Delta I = 1/2$ rule in the $K \rightarrow 2\pi$ decay.

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